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A NON-EXISTENCE RESULT FOR FINITE PROJECTIVE PLANES IN LENZ–BARLOTTI CLASS I.4

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Let Π be a projective plane of order n in Lenz–Barlotti class I.4, and assume that n is a multiple of 3. Then either $n=3$ or n is a multiple of 9.

We shall assume that the reader is familiar with the basic theory of finite projective planes, in particular with the notions of elations, homologies, (p,L)-transitivity and the idea of the Lenz–Barlotti classification. For background, we refer the reader to Dembowski [2], Hughes and Piper [6], Pickert [8] and Yaqub [11].

In the Lenz–Barlotti classification, collineation groups of projective planes are classified according to the configuration F formed by the point-line pairs (p,L) for which the given group G is (p,L)-transitive; in the special case $G=\operatorname{Aut}\Pi$, one speaks of the *Lenz–Barlotti class* of Π. For a group G of type I.4, F consists of the vertices and the opposite sides of a triangle. Equivalently, G is a (necessarily abelian) quasiregular collineation group of type (g) in the Dembowski–Piper classification [3]. For a proof of this fact, more background and historical references, we refer the reader to our recent systematic treatment of planes in class I.4 [5].

The only known finite planes admitting a group of type I.4 are the Desarguesian planes. Any other example would necessarily be in Lenz–Barlotti class I.4, and it is widely conjectured that there are no finite planes in this class (but there do exist infinite examples). We here add another bit of evidence for this conjecture by proving the following new non-existence result.

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Theorem 1. *Let Π be a projective plane of order n in Lenz–Barlotti class I.4, and assume that n is a multiple of 3. Then either $n=3$ or n is a multiple of 9.*

Proof. As Π admits a group of type I.4, it may be represented by an *abelian neo-difference set*, see [5]. Using group ring notation – a standard approach in the study of any type of difference set, see [1] or our survey [4] for background and notation – D is a subset of a group $G=X\times X$ satisfying the equation

$$DD^{(-1)} = n + G - U_1 - U_2 - U_3 \tag{1}$$

in $\mathbb{Z}G$; here X is a (multiplicatively written) abelian group of order $n-1$ and $U_1 = X\times\{1\}$, $U_2=\{1\}\times X$, $U_3=\{(\xi,\xi):\xi\in X\}$. Thus every element γ not in the union of the three *forbidden subgroups* U_1, U_2, and U_3 has a unique "difference representation" $\gamma=\delta\varepsilon^{-1}$ with $\delta,\varepsilon\in D$. Note that $|D|=n-2$ and hence, for $i=1,2,3$, there is exactly one coset of U_i which misses D, whereas every other coset intersects D uniquely, as no element in U_i has a difference representation from D. As shown in [5], we may assume that both U_1 and U_2 are disjoint from D and that the unique coset of U_3 missing D is $U_3(1,\theta)$, where θ is an (in fact, the unique) involution in X if n is odd and $\theta=1$ otherwise.

As 3 divides n, it is a multiplier for D, that is, $D^{(3)}=D$ in $\mathbb{Z}G$. It is well-known and easy to see that $D^3\equiv D^{(3)} \pmod 3$, and thus we may write $D^3-D=3A$ for some $A\in\mathbb{Z}G$; then also $(D^{(-1)})^3-D^{(-1)}=3A^{(-1)}$. This implies the following congruence modulo 9:

$$0\equiv(3A)(3A^{(-1)})\equiv(DD^{(-1)})^3+DD^{(-1)}\Big[1-D^2-(D^{(-1)})^2\Big] \pmod 9.$$

Writing $N=U_1+U_2+U_3$ and applying (1), we obtain

$$\begin{aligned} &n\Big[D^2+(D^{(-1)})^2\Big] \\ &\quad\equiv(n+G-N)^3+(n+G-N)-(G-N)\Big[D^2+(D^{(-1)})^2\Big] \pmod 9. \end{aligned}$$

Using $3\,|\,n$, it is not difficult but definitely somewhat tedious to show that the preceding equation reduces to

$$n\Big[D^2+(D^{(-1)})^2\Big]\equiv n+nG-nN \pmod 9. \tag{2}$$

For the convenience of the reader, we list a few auxiliary results which help establishing (2):

$$DG=(n-2)G;$$

$$\begin{aligned}
DU_i &= G - U_i \ \text{ for } i = 1,2 \quad \text{and} \quad DU_3 = G - U_3(1,\theta);\\
D^2U_i &= (n-3)G + U_i \quad \text{for } i = 1,2,3;\\
D^2N &\equiv N \pmod 9;\\
-(G-N)D^2 &\equiv 4(n-1)G + N \pmod 9;\\
N^2 &= (n-1)N + 6G;\\
(n+G-N)^2 &\equiv 4G - (n+1)N \pmod 9;\\
(n+G-N)^3 &\equiv -(n+2)G - (n+1)N \pmod 9.
\end{aligned}$$

We now assume that $n = 3m$ for some integer m not divisible by 3. Then equation (2) implies

$$D^2 + (D^{(-1)})^2 \equiv 1 + G - N \text{ in } \mathbb{Z}_3G. \tag{3}$$

Observe that all terms d_id_j in

$$D^2 = \sum_{i=1}^{n-2} d_i^2 + 2\sum_{i<j} d_id_j$$

are distinct, since $d_id_j = d_kd_l$ would imply $d_id_k^{-1} = d_ld_j^{-1}$, contradicting the uniqueness of difference representations from D; of course, the analogous assertion also holds for $(D^{(-1)})^2$. Hence the left-hand side of (3) contains at most $2(n-2) + \binom{n-2}{2}$ group elements with coefficient 1, whereas the right-hand side of (3) contains exactly $(n-1)^2 - 1 - 3(n-2)$ such group elements. Therefore

$$2(n-2) + \frac{n^2 - 5n + 6}{2} \geq n^2 - 5n + 6$$

which implies $n \leq 7$ and hence $n = 3$, since $n = 6$ is well-known to be impossible. ∎

Remark 2. The reader may have noted that equation (3) is analogous to the Wilbrink equation [10] for planar difference sets; indeed, the proof presented above resembles the known purely computational proofs for Wilbrink's theorem and Pott's [9] corresponding result for affine difference sets, see [4] and [7]. It seems not unlikely that, more generally, the following analogue of the Wilbrink equation holds for any abelian neo-difference set D of any order n which is exactly divisible by a prime p:

$$D^{p-1} + (D^{(-1)})^{p-1} = 1 + G - N \text{ in } \mathbb{Z}_pG. \tag{4}$$

Clearly, some equation of this type has to hold, but a computational proof along the lines above would certainly be extremely unpleasant. We note that nobody has yet succeeded in using the Wilbrink approach to obtain any non-existence result for a prime $p \geq 5$ in either the planar or the affine case: it seems that dealing with high powers of D is just hopeless. In view of these considerations, we decided that it would not be worth the effort to try and prove a more general formula such as equation (4).

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